WEAK SEPARATION AXIOMS VIA \( \Omega - \text{OPEN SET AND} \quad \Omega - \text{CLOSURE OPERATOR} \)

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**ABSTRACT**

In this paper we introduce a new type of weak separation axioms with some related theorems and show that they are equivalent with these in [1].

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Sober \( \omega - R_0, \quad \omega - R_0 \text{ space,} \quad \omega - R_1 \text{ space.} \)

**1. INTRODUCTION**

In this article let us prepare the background of the subject. Throughout this paper , \((X, T)\) stands for topological space. Let \(A\) be a subset of \(X\). A point \(x\) in \(X\) is called *condensation* point of \(A\) if for each \(U\) in \(T\) with \(x\) in \(U\), the set \(U \cap A\) is uncountable [2]. In 1982 the \(\omega - \text{closed set was first introduced by Hdeib [2], and he defined it as:} \quad A \text{ is } \omega - \text{closed if it contains all its condensation points and the } \omega - \text{open set is the complement of the } \omega - \text{closed set. It is not hard to prove: any open set is } \omega - \text{open. Also we would like to say that the collection of all } \omega - \text{open subsets of } X \text{ forms topology on } X. \text{ The closure of } A \text{ will be denoted by } cl(A), \text{ while the intersection of all } \omega - \text{closed sets in } X \text{ which containing } A \text{ is called the } \omega - \text{closure of } A, \text{ and will denote by } cl_\omega(A). \text{ Note that } cl_\omega(A) \subset cl(A). \text{ In 2005 Caldas, et al. [3] introduced some weak separation axioms by utilizing the notions of } \delta - \text{pre } \omega - \text{open sets and } \delta - \text{pre } \omega - \text{closure. In this paper we use Caldas, et al. [3] definitions to introduce new spaces by using the } \omega - \text{open sets defined by Hdeib [2], we call it } \omega - R_i - \text{Spaces } i = 0,1,2, \text{ and we show that } \omega - R_0 , \omega^* - T_1 \text{ space and } \omega - \text{symmetric space are equivalent.} \text{ For our main results we need the following definitions and results:} \text{ Definition-1.1.} \text{ Noiri, et al. [4] A space } (X,T) \text{ is called a } door space \text{ if every subset of } X \text{ is either open or closed.}
Definition-1.2.

Hadi [1] The topological space \( X \) is called \( \omega^* - T_1 \) space if and only if, for each \( x \neq y \in X \), there exist \( \omega \) open sets \( U \) and \( V \), such that \( x \in U, y \notin U \), and \( y \in V, x \notin V \).

Lemma-1.3.

Hadi [1] The topological space \( X \) is \( \omega^* - T_1 \) if and only if for each \( x \in X \), \( \{x\} \) is \( \omega \) closed set in \( X \).

Definition-1.4.

Hadi [1] The topological space \( X \) is called \( \omega^* - T_2 \) space if and only if, for each \( x \neq y \in X \), there exist two disjoint \( \omega \) open sets \( U \) and \( V \) with \( x \in U \) and \( y \in V \).

For our main result we need the following property of \( \omega \) closure of a set:

Proposition-1.5.

Let \( \{A_j, \lambda \in \Lambda\} \) be a family of subsets of the topological space \( (X, T) \), then

1. \( cl_\omega(n_{\lambda \in \Lambda} A_\lambda) \subseteq n_{\lambda \in \Lambda} cl_\omega(A_\lambda) \).
2. \( U_{\lambda \in \Lambda} cl_\omega(A_\lambda) \subseteq cl_\omega(U_{\lambda \in \Lambda} A_\lambda) \).

Proof:

1. It is clear that \( n_{\lambda \in \Lambda} A_\lambda \subseteq A_\lambda \) for each \( \lambda \in \Lambda \). Then by (4) of Theorem 1.5.3 in Hadi [1], we have
   \( cl_\omega(n_{\lambda \in \Lambda} A_\lambda) \subseteq cl_\omega(A_\lambda) \) for each \( \lambda \in \Lambda \). Therefore
   \( cl_\omega(n_{\lambda \in \Lambda} A_\lambda) \subseteq n_{\lambda \in \Lambda} cl_\omega(A_\lambda) \).
   Note that the opposite direction is not true. For example consider the usual topology \( T \) for \( \mathbb{R} \), If
   \( A_i = (0, \frac{1}{i}), i = 1, 2, \ldots \), and \( \cap_{i \in \mathbb{N}} cl_\omega(A_i) = \{0\} \). But \( cl_\omega(\cap_{i \in \mathbb{N}} A_i) = cl_\omega(\emptyset) = \emptyset \). Therefore
   \( n_{\lambda \in \Lambda} cl_\omega(A_\lambda) \nsubseteq cl_\omega(n_{\lambda \in \Lambda} A_\lambda) \).

2. Since \( A_\lambda \subseteq U_{\lambda \in \Lambda} A_\lambda \) for each \( \lambda \in \Lambda \). Then by (4) of Theorem 1.5.3 in Hadi [1], we get
   \( cl_\omega(A_\lambda) \subseteq cl_\omega(U_{\lambda \in \Lambda} A_\lambda) \), for each \( \lambda \in \Lambda \). Hence \( U_{\lambda \in \Lambda} cl_\omega(A_\lambda) \subseteq cl_\omega(U_{\lambda \in \Lambda} A_\lambda) \).
   Note that the opposite direction is not true. For example consider the usual topology \( T \) for \( \mathbb{R} \), If
   \( A_i = \{1\}, i = 1, 2, \ldots \), \( cl_\omega(A_i) = \{1\} \) and \( \cup_{i \in \mathbb{N}} cl_\omega(A_i) = \{1, \frac{1}{2}, \frac{1}{3}, \ldots \} \). But \( cl_\omega(\cup_{i \in \mathbb{N}} A_i) = \{1, \frac{1}{2}, \frac{1}{3}, \ldots , 0\} \). Thus \( cl_\omega(\cup_{\lambda \in \Lambda} A_\lambda) \nsubseteq \cup_{\lambda \in \Lambda} cl_\omega(A_\lambda) \).  

2. \( \Omega - R_1 - \text{SPACES, FOR } i = 0, 1 \)

In this section we introduce some types of weak separation axioms by utilizing the \( \omega \) open sets defined in Hdeib [2].

Definition-2.1.

Let \( A \subset (X, T) \), then the \( \omega \) kernel of \( A \) denoted by \( \omega - ker(A) \) is the set
\( \omega - ker(A) = \cap \{O, \text{where } O \text{ is an } \omega \text{-open set in } (X, T) \text{ containing } A\} \).

Proposition-2.2.

Let \( A \subset (X, T) \), and \( x \in X \). Then
\[ \omega - ker(A) = \{ x \in X : cl_\omega(\{x\}) \cap A \neq \emptyset \}. \]

**Proof:**

Let \( A \) be a subset of \( X \), and \( x \in \omega - ker(A) \), such that \( cl_\omega(\{x\}) \cap A = \emptyset \). Then \( x \notin X \setminus cl_\omega(\{x\}) \), which is an \( \omega \) -open set containing \( A \). This contradicts \( x \in \omega - ker(A) \). So \( cl_\omega(\{x\}) \cap A \neq \emptyset \).

Then let \( x \in X \), be a point satisfied \( cl_\omega(\{x\}) \cap A \neq \emptyset \). Assume \( x \notin \omega - ker(A) \), then there exists an \( \omega \) -open set \( G \) containing \( A \) but not \( x \). Let \( y \in cl_\omega(\{x\}) \cap A \). Hence \( G \) is an \( \omega \) -open set containing \( y \) but not \( x \). This contradicts \( cl_\omega(\{x\}) \cap A \neq \emptyset \). So \( x \in \omega - ker(A) \).

**Definition-2.3.**

A topological space \((X, T)\) is said to be sober \( \omega - R_0 \) if \( \cap_{x \in X} cl_\omega(\{x\}) = \emptyset \).

**Theorem-2.4.**

A topological space \((X, T)\) is sober \( \omega - R_0 \) if and only if \( \omega - ker(\{x\}) \neq X \) for each \( x \in X \).

**Proof:**

Suppose that \((X, T)\) is sober \( \omega - R_0 \). Assume there is a point \( y \in X \), with \( \omega - ker(\{y\}) = X \). Let \( x \in X \), then \( x \in V \) for any \( \omega \) -open set \( V \) containing \( y \), so \( y \in cl_\omega(\{x\}) \) for each \( x \in X \). This implies \( y \in \cap_{x \in X} cl_\omega(\{x\}) \), which is a contradiction with \( \cap_{x \in X} cl_\omega(\{x\}) = \emptyset \).

Now suppose \( \omega - ker(\{x\}) \neq X \) for every \( x \in X \). Assume \( X \) is not sober \( \omega - R_0 \), it mean there is \( y \) in \( X \) such that \( y \in \cap_{x \in X} cl_\omega(\{x\}) \), then every \( \omega \) -open set containing \( y \) must contain every point of \( X \). This implies that \( X \) is the unique \( \omega \) -open set containing \( y \). Therefore \( \omega - ker(\{y\}) = X \), which is a contradiction with our hypothesis. Hence \((X, T)\) is sober \( \omega - R_0 \).

**Definition-2.5.**

A map \( f : X \rightarrow Y \) is called \( \omega - closed \), if the image of every \( \omega \) -closed subset of \( X \) is \( \omega \) -closed in \( Y \).

**Proposition-2.6.**

If \( X \) is a space, \( f \) is a map defined on \( X \) and \( A \subseteq X \), then \( cl_\omega(f(A)) \subseteq f(cl_\omega(A)) \).

**Proof:**

We have \( A \subseteq cl_\omega(A) \), then \( f(A) \subseteq f(cl_\omega(A)) \). This implies \( cl_\omega(f(A)) \subseteq cl_\omega(f(cl_\omega(A))) = f(cl_\omega(A)) \). Hence \( cl_\omega(f(A)) \subseteq f(cl_\omega(A)) \).

**Theorem-2.7.**

If \( f : X \rightarrow Y \) is one to one \( \omega \) -closed map and \( X \) is sober \( \omega - R_0 \), then \( Y \) is sober \( \omega - R_0 \).

**Proof:**

From Proposition 1.5, we have
\[ \cap_{y \in Y} cl_\omega(\{y\}) \subseteq \cap_{x \in X} cl_\omega(\{f(x)\}) \subseteq \cap_{x \in X} f(cl_\omega(\{x\})) \]
\[ = f(\cap_{x \in X} cl_\omega(\{x\})) \]
\[ = f(\emptyset) = \emptyset. \]
Thus $Y$ is sober $\omega - R_0$

**Definition-2.8.**

A topological space $(X, T)$ is called $\omega - R_0$ if every $\omega -$open set contains the $\omega -$closure of each of its singletons.

**Theorem-2.9.**

The topological door space is $\omega - R_0$ if and only if it is $\omega^* - T_1$.

**Proof:**

Let $x, y$ are distinct points in $X$. Since $(X,T)$ is door space so that for each $x$ in , $\{x\}$ is open or closed.

i. 1. When $\{x\}$ is open, hence $\omega -$open set in $X$. Let $V = \{x\}$ , then $x \in V$, and $y \notin V$. Therefore since $(X,T)$ is $\omega - R_0$ space, so that $cl_{\omega}((x)) \subseteq V$. Then $x \notin X\setminus V$, while $y \in X\setminus V$, where $X\setminus V$ is an $\omega -$open subset of $X$.

2. Whenever $\{x\}$ is closed, hence it is $\omega -$closed, $y \in X\setminus \{x\}$, and $X\setminus \{x\}$ is $\omega -$open set in $X$. Then since $(X,T)$ is $\omega - R_0$ space, so that $cl_{\omega}((y)) \subseteq X\setminus \{x\}$. Let $V = X\setminus cl_{\omega}((y))$, then $x \in V$, but $y \notin V$, and $V$ is an $\omega -$open set in $X$. Thus we obtain $(X,T)$ is $\omega^* - T_1$.

ii. For the other direction assume $(X,T)$ is $\omega^* - T_1$, and let $V$ be an $\omega -$open set of $X$, and $x \in V$. For each $y \in X\setminus V$, there is an $\omega -$open set $V_y$ such that $x \notin V_y$, but $y \in V_y$. So $cl_{\omega}((x)) \cap V_y = \emptyset$, which is true for each $y \in X\setminus V$. Therefore $cl_{\omega}((x)) \cap (\bigcup_{y \in X\setminus V} V_y) = \emptyset$. Then since $y \in V_y$, $X\setminus V \subseteq \bigcup_{y \in X\setminus V} V_y$, and $cl_{\omega}((x)) \subseteq V$. Hence $(X,T)$ is $\omega - R_0$.

**Definition-2.10.**

A topological space $(X,T)$ is $\omega -$symmetric if for $x$ and $y$ in the space $X$, $x \in cl_{\omega}((y))$ implies $y \in cl_{\omega}((x))$.

**Proposition-2.11.**

Let $X$ be a door $\omega -$symmetric topological space . Then for each $x \in X$, the set $\{x\}$ is $\omega -$closed.

**Proof:**

Let $x \neq y \in X$, since $X$ is a door space so $\{y\}$ is open or closed set in $X$. When $\{y\}$ is open, so it is $\omega -$open, let $V_y = \{y\}$. Whenever $\{y\}$ is $\omega -$closed , $x \notin \{y\} = cl_{\omega}((y))$. Since $X$ is $\omega -$symmetric we get $y \notin cl_{\omega}((x))$. Put $V_x = X\setminus cl_{\omega}((x))$, then $x \notin V_x$ and $y \in V_x$, and $V_x$ is $\omega -$open set in $X$. Hence we get for each $y \in X\setminus \{x\}$ there is an $\omega -$open set $V_y$ such that $x \notin V_y$ and $y \in V_y$. Therefore $X\setminus \{x\} = \bigcup_{y \in X\setminus \{x\}} V_y$ is $\omega -$open, and $\{x\}$ is $\omega -$closed.

**Proposition-2.12.**

Let $(X,T)$ be $\omega^* - T_1$ topological space, then it is $\omega -$symetric space.

**Proof:**

Let $x \neq y \in X$. Assume $y \notin cl_{\omega}((x))$, then since $X$ is $\omega - T_1$ there is an open set $U$ containing $x$ but not $y$, so $x \notin cl_{\omega}((y))$. This completes the proof.
Theorem-2.13.

The topological door space is $\omega -$ symmetric if and only if it is $\omega^* - T_1$.

Proof:
Let $(X, T)$ be a door $\omega -$ symmetric space. Then using Proposition 2.11 for each $x \in X$, $\{x\}$ is $\omega -$ closed set in $X$. Then Lemma 1.3, we get that $(X, T)$ is $\omega^* - T_1$. On the other hand, assume $(X, T)$ is $\omega^* - T_1$, then directly by Proposition 2.12, $(X, T)$ is $\omega -$ symmetric space.


Let $(X, T)$ be a topological door space, then the following are equivalent:
1. $(X, T)$ is $\omega - R_0$ space.
2. $(X, T)$ is $\omega^* - T_1$ space.
3. $(X, T)$ is $\omega -$ symmetric space.

Proof:
The proof follows immediately from Theorem 2.9 and Theorem 2.13.

Corollary-2.15.

If $(X, T)$ is a topological door space, then it is $\omega - R_0$ space if and only if for each $x \in X$, the set $\{x\}$ is $\omega -$ closed set.

Proof:
We can prove this corollary by using Corollary 2.14 and Lemma 1.3.

Theorem-2.16.

Let $(X, T)$ be a topological space contains at least two points. If $X$ is $\omega - R_0$ space, then it is sober $\omega - R_0$ space.

Proof:
Let $x$ and $y$ are two distinct points in $X$. Since $(X, T)$ is $\omega - R_0$ space so by Theorem 2.8 it is $\omega^* - T_1$. Then Lemma 1.3 implies $cl_{\omega}(\{x\}) = \{x\}$ and $cl_{\omega}(\{y\}) = \{y\}$. Therefore $\cap_{p \in \{x, y\}} cl_{\omega}(\{p\}) \subset cl_{\omega}(\{x\} \cap cl_{\omega}(\{y\})) = \{x\} \cap \{y\} = \emptyset$. Hence $(X, T)$ is sober $\omega - R_0$ space.

Definition-2.17.

A topological door space $(X, T)$ is said to be $\omega - R_1$ space if for $x$ and $y$ in $X$, with $cl_{\omega}(\{x\}) \neq cl_{\omega}(\{y\})$, there are disjoint $\omega -$ open set $U$ and $V$ such that $cl_{\omega}(\{x\}) \subset U$, and $cl_{\omega}(\{y\}) \subset V$.

Theorem-2.18.

The topological door space is $\omega - R_1$ if and only if it is $\omega^* - T_2$ space.

Proof:
Let $x$ and $y$ be two distinct points in $X$. Since $X$ is door space so for each $x$ in $X$, The set $\{x\}$ is open or closed.

i. If $\{x\}$ is open. Since $\{x\} \cap \{y\} = \emptyset$, then $\{x\} \cap cl_{\omega}(\{y\}) = \emptyset$. Thus $cl_{\omega}(\{x\}) \neq cl_{\omega}(\{y\})$. 

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ii. Whenever \( \{x\} \) is closed, so it is \( \omega \)-closed and \( cl_\omega(\{x\}) \cap \{y\} = \{x\} \cap \{y\} = \emptyset \). Therefore \( cl_\omega(\{x\}) \neq cl_\omega(\{y\}) \). We have \( (X, T) \) is \( \omega - R_1 \) space, so that there are disjoint \( \omega \)-open sets \( U \) and \( V \) such that \( x \in cl_\omega(\{x\}) \subset U \), and \( y \in cl_\omega(\{y\}) \subset V \), so \( X \) is \( \omega^* - T_2 \) space.

For the opposite side let \( x \) and \( y \) be any points in \( X \), with \( cl_\omega(\{x\}) \neq cl_\omega(\{y\}) \). Since every \( \omega^* - T_2 \) space is \( \omega^* - T_1 \) space so by (3) of Theorem 2.2.15 \( cl_\omega(\{x\}) = \{x\} \) and \( cl_\omega(\{y\}) = \{y\} \), this implies \( x \neq y \). Since \( X \) is \( \omega^* - T_2 \) there are two disjoint \( \omega \)-open sets \( U \) and \( V \) such that \( cl_\omega(\{x\}) = \{x\} \subset U \), and \( cl_\omega(\{y\}) = \{y\} \subset V \). This proves \( X \) is \( \omega - R_1 \) space.

**Corollary-2.19.**

Let \( (X, T) \) be a topological door space. Then if \( X \) is \( \omega - R_1 \) space then it is \( \omega - R_0 \) space.

**Proof:**

Let \( X \) be an \( \omega - R_1 \) door space. Then by Theorem 2.17 \( X \) is \( \omega^* - T_2 \) space. Then since every \( \omega^* - T_2 \) space is \( \omega^* - T_1 \), so that by Theorem 2.9, \( X \) is \( \omega - R_0 \) space.

**REFERENCES**


