ON THE STARLIKENESS FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT
Let \( f(z) \) be an analytic function in the open unit disk \( U \) normalized with \( f(0) = 0 \) and \( f'(0) = 1 \). In this paper, the starlikeness for \( f(z) \) is discussed.

Key Words: Analytic functions; Starlike function; Close-to-convex functions.

2000 Mathematical Subject Classification: 30C45.

INTRODUCTION
Let \( H \) be the class of analytic functions in \( U = \{ z \in \mathbb{C} : |z| < 1 \} \), and \( A \) be the subclass of \( H \) consisting of functions of the form

\[
f(z) = z + a_2 z^2 + a_3 z^3 + \cdots , z \in U.
\]

A function \( f(z) \in A \) is said to be starlike of order \( \alpha (0 \leq \alpha < p) \) in \( U \) (see Robertson (1936)), that is, \( f(z) \in S^*(\alpha) \), if and only if

\[
\text{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \alpha , 0 \leq \alpha < 1, z \in U
\]

with \( S^*_1(0) := S^* \).
Similarly, a function $f(z) \in A$ is said to be convex of order $\alpha (0 \leq \alpha < 1)$ in $U$ (see Robertson (1936)), that is, $f(z) \in K(\alpha)$, if and only if
\[
\text{Re}(1 + \frac{zf''(z)}{f'(z)}) > \alpha, 0 \leq \alpha < 1, z \in U
\] (3)
with $K(0) = K$.

By the definitions for the classes $S^*(\alpha)$ and $K(\alpha)$, we know that $f(z) \in K(\alpha)$ if and only if $f(z) \in S^*(\alpha)$. Marx (1932/33) and Strohhäcker (1933) showed that $f(z) \in K(0)$ implies $f(z) \in S^*(1/2)$.

Several results appeared previously about sufficient conditions of starlikeness (see (Nunokawa et al., 2012; Sokol, 2012)). In this paper, With the help of two inequality, the starlikeness for $f(z)$ is discussed.

The Main Results

Lemma 2.1. (see Nunokawa et al. (2012)) Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic in the unit disc $U$ and $\alpha (0 < \alpha \leq 1/2)$ be a positive real number. Then suppose that there exists a point $z_0 \in U$ such that
\[
\text{Re}(z) > \alpha \text{ for } |z| < |z_0|
\] (4)
and
\[
\text{Re}(z_0) = \alpha, p(z_0) \neq \alpha.
\] (5)
Then we have
\[
\frac{z_0p'(z_0)}{p(z_0)} \leq \frac{\alpha}{2(1-\alpha)}.
\] (6)

By using Lemma 2.1, we first prove the following Theorem.

Theorem 2.1. Let $f(z) \in A$, and $\alpha (0 < \alpha \leq 1/2)$ be a positive real number. Suppose
\[
\frac{zf'(z)}{f(z)} \neq \alpha
\] (7)
and
Then we have $f(z) \in S^*(\alpha)$.

**Proof.** Let

$$p(z) = \frac{zf'(z)}{f(z)},$$

then $p(z)$ is analytic in $U$ and $p(0) = 1$. Suppose that there exists a point $z_0 \in U$ which satisfies the conditions (4) and (5) of Lemma 2.1.

Now using (9), it follows that

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)}.$$

Since the function $p(z)$ and the point $z_0$ satisfy all conditions Lemma 2.1, therefore in view of (6) and (10) gives

$$Re\left(1 + \frac{zf''(z_0)}{f'(z_0)}\right) = Re\left(\frac{zp'(z_0)}{p(z_0)} + p(z_0)\right).$$

This is a contradiction and therefore proof of the Theorem 2.1 is completed.

**Lemma 2.2.** (see [6]) Let $p(z) = 1 + c_1z + c_2z^2 + \cdots$ be analytic in the unit disc $U$ and $\alpha(1/2 < \alpha < 1)$ be a positive real number. Then suppose that there exists a point $z_0 \in U$ such that

$$Re(z) > \alpha \text{ for } |z| < |z_0|$$

and

$$Re(z_0) = \alpha, p(z_0) \neq \alpha.$$

Then we have

$$\frac{zp'(z_0)}{p(z_0)} \leq -\frac{1-\alpha}{2\alpha}.$$

By using Lemma 2.2, we can prove the following Theorem.
Theorem 2.2. Let \( f(z) \in A \), and \( \alpha (1/2 < \alpha < 1) \) be a positive real number. Suppose
\[
\frac{zf'(z)}{f(z)} \neq \alpha \quad (15)
\]
and
\[
Re(1 + \frac{zf''(z)}{f'(z)}) > Re(\frac{zf'(z)}{f(z)}) - \frac{1 - \alpha}{2\alpha} \quad (16)
\]
Then we have \( f(z) \in S^*(\alpha) \).

Proof. Let
\[
p(z) = \frac{zf'(z)}{f(z)}, \quad (17)
\]
then \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). Suppose that there exists a point \( z_0 \in U \) which satisfies the conditions (12) and (13) of Lemma 2.2.

Now using (17), it follows that
\[
1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)} \quad (18)
\]
Since the function \( p(z) \) and the point \( z_0 \) satisfy all conditions Lemma 2.2, therefore in view of (14) and (18) gives
\[
Re(1 + \frac{zf''(z_0)}{f'(z_0)}) = Re(\frac{zp'(z_0)}{p(z_0)} + p(z_0)) \quad (19)
\]
This is a contradiction and therefore proof of the Theorem 2.2 is completed.

REFERENCES