ON THE STARLIKENESS FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT
Let $f(z)$ be an analytic function in the open unit disk $U$ normalized with $f(0) = 0$ and $f'(0) = 1$. In this paper, the starlikeness for $f(z)$ is discussed.

Key Words: Analytic functions; Starlike function; Close-to-convex functions.

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INTRODUCTION

Let $H$ be the class of analytic functions in $U = \{ z \in \mathbb{C} : |z| < 1 \}$, and $A$ be the subclass of $H$ consisting of functions of the form

\[ f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, z \in U. \quad (1) \]

A function $f(z) \in A$ is said to be starlike of order $\alpha (0 \leq \alpha < p)$ in $U$ (see Robertson (1936)), that is, $f(z) \in S^*(\alpha)$, if and only if

\[ \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, 0 \leq \alpha < 1, z \in U \quad (2) \]

with $S^*_1(0) := S^*$.
Similarly, a function $f(z) \in A$ is said to be convex of order $\alpha (0 \leq \alpha < 1)$ in $U$ (see Robertson (1936)), that is, $f(z) \in K(\alpha)$, if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, 0 \leq \alpha < 1, z \in U$$

(3)

with $K(0) = K$.

By the definitions for the classes $S^*(\alpha)$ and $K(\alpha)$, we know that $f(z) \in K(\alpha)$ if and only if $f(z) \in S^*(\alpha)$. Marx (1932/33) and Strohhäcker (1933) showed that $f(z) \in K(0)$ implies $f(z) \in S^*(1/2)$.

Several results appeared previously about sufficient conditions of starlikeness (see Nunokawa et al., 2012; Sokol, 2012). In this paper, With the help of two inequality, the starlikeness for $f(z)$ is discussed.

The Main Results

**Lemma 2.1.** (see Nunokawa et al. (2012)) Let $p(z) = 1 + c_1z + c_2z^2 + \cdots$ be analytic in the unit disc $U$ and $\alpha (0 < \alpha \leq 1/2)$ be a positive real number. Then suppose that there exists a point $z_0 \in U$ such that

$$\text{Rep}(z) > \alpha \text{ for } |z| < |z_0|$$

(4)

and

$$\text{Rep}(z_0) = \alpha, p(z_0) \neq \alpha.$$  

(5)

Then we have

$$\frac{z_0p'(z_0)}{p(z_0)} \leq -\frac{\alpha}{2(1-\alpha)}.$$  

(6)

By using Lemma 2.1, we first prove the following theorem.

**Theorem 2.1.** Let $f(z) \in A$, and $\alpha (0 < \alpha \leq 1/2)$ be a positive real number. Suppose

$$\frac{zf'(z)}{f(z)} \neq \alpha$$

(7)

and
\[ Re(1 + \frac{zf''(z)}{f'(z)}) > Re(\frac{zf'(z)}{f(z)}) - \frac{\alpha}{2(1 - \alpha)}. \]  

Then we have \( f(z) \in S^*(\alpha) \).

**Proof.** Let

\[ p(z) = \frac{zf'(z)}{f(z)}, \]  

then \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). Suppose that there exists a point \( z_0 \in U \) which satisfies the conditions (4) and (5) of Lemma 2.1.

Now using (9), it follows that

\[ 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)}. \]  

Since the function \( p(z) \) and the point \( z_0 \) satisfy all conditions Lemma 2.1, therefore in view of (6) and (10) gives

\[ Re(1 + \frac{z_0f''(z_0)}{f'(z_0)}) = Re(\frac{zp'(z_0)}{p(z_0)} + p(z_0)). \]  

This is a contradiction and therefore proof of the Theorem 2.1 is completed.

**Lemma 2.2.** (see [6]) Let \( p(z) = 1 + c_1z + c_2z^2 + \cdots \) be analytic in the unit disc \( U \) and \( \alpha(1/2 < \alpha < 1) \) be a positive real number. Then suppose that there exists a point \( z_0 \in U \) such that

\[ Rep(z) > \alpha \text{ for } |z| < |z_0| \]  

and

\[ Rep(z_0) = \alpha, p(z_0) \neq \alpha. \]  

Then we have

\[ \frac{z_0p'(z_0)}{p(z_0)} \leq -\frac{1-\alpha}{2\alpha}. \]  

By using Lemma 2.2, we can prove the following Theorem.
**Theorem 2.2.** Let $f(z) \in A$, and $\alpha(1/2 < \alpha < 1)$ be a positive real number. Suppose
\[
\frac{zf''(z)}{f'(z)} \neq \alpha
\]  
(15)

and
\[
Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > Re\left(\frac{zf''(z)}{f'(z)} \cdot \frac{1 - \alpha}{2\alpha}\right).
\]  
(16)

Then we have $f(z) \in S^*(\alpha)$.

**Proof.** Let
\[
p(z) = \frac{zf'(z)}{f(z)},
\]  
(17)

then $p(z)$ is analytic in $U$ and $p(0) = 1$. Suppose that there exists a point $z_0 \in U$ which satisfies the conditions (12) and (13) of Lemma 2.2.

Now using (17), it follows that
\[
1 + \frac{zf''(z)}{f'(z)} = \frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)},
\]  
(18)

Since the function $p(z)$ and the point $z_0$ satisfy all conditions Lemma 2.2, therefore in view of (14) and (18) gives
\[
Re\left(1 + \frac{zf''(z_0)}{f'(z_0)}\right) = Re\left(\frac{zp'(z_0)}{p(z_0)} + p(z_0)\right).
\]  
(19)

This is a contradiction and therefore proof of the Theorem 2.2 is completed.

**REFERENCES**